A model for Clearpool interest rates

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Abstract

Few simple models to describe the interest rate as function of the utilization ratio are presented. The rationale behind them is to have a relatively simple and continuous mathematical function with domain $\mathcal{D} = [0, 1]$ that

(i) has given values at $x = 0$ and $x = 1$;
(ii) approaches the domain boundaries with as-small-as-possible derivative;
(iii) has exactly one global minimum in the domain at x_m;
(iv) it is monotonically decreasing and increasing for $x < x_m$ and $x > x_m$, respectively;
(v) it is *pretty* flat around the minimum.

Note that property *(iv)* can be expressed as not having any other extremum within $\mathcal{D}$.

To stay as general as possible, parameters have been introduced to be able to adjust the final curve according to own needs. However, it is desired to already impose two further properties and namely that

(vi) $x_m \approx 0.8$ and
(vii) $f(0) < f(1)$.

1 Using polynomials

Using polynomials would be advantageous at later stages in terms of numerical cost. The most naive approach is to impose properties *(i)* to *(iii)* as a set of 5 conditions on a fourth order polynomial. Considering

$$p(x) \equiv a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 \tag{1}$$

and imposing

$$p(0) = A\,, \qquad p(1) = B \qquad \text{and} \qquad p'(0) = p'(1) = p'(x_m) = 0\,,$$

it is possible to determine all the coefficients a_i, with $i \in \{0, 1, 2, 3, 4\}$, obtaining

$$a_0 = 1\,, \quad a_1 = 0\,, \quad a_2 = \frac{6\,(A-B)\,x_m}{1-2\,x_m}\,, \quad a_3 = \frac{4\,(A-B)\,(1+x_m)}{2\,x_m-1}\,, \quad a_4 = \frac{3\,(A-B)}{1-2\,x_m}\,. \tag{2}$$

which plugged into eq. (1) completely define $p(x)$. The last imposed condition $p'(x_m) = 0$ only requires that p has an extremum at x_m, but not that this is a minimum. It is possible to show that

$$p''(x_m) = \frac{12\,x_m\,(1-x_m)\,(A-B)}{2\,x_m-1}\,,$$

which implies that $x = x_m$ is a minimum if and only if

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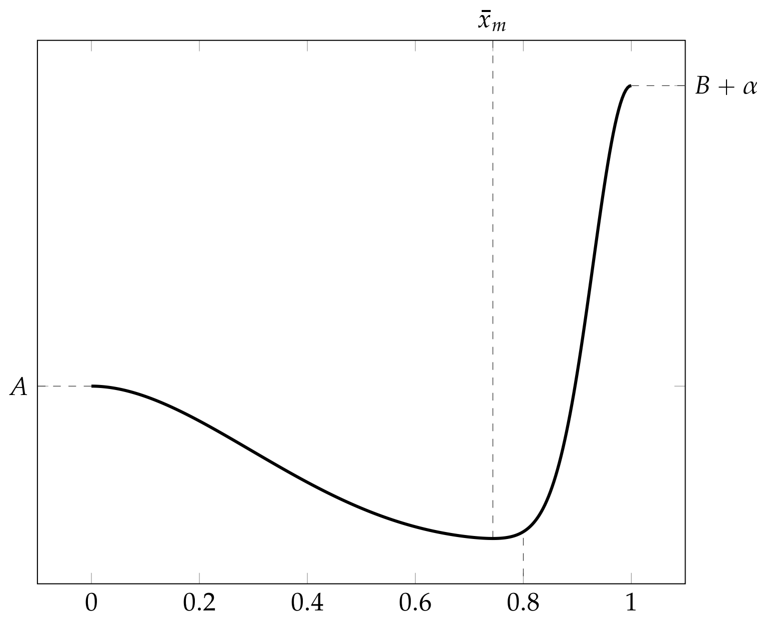

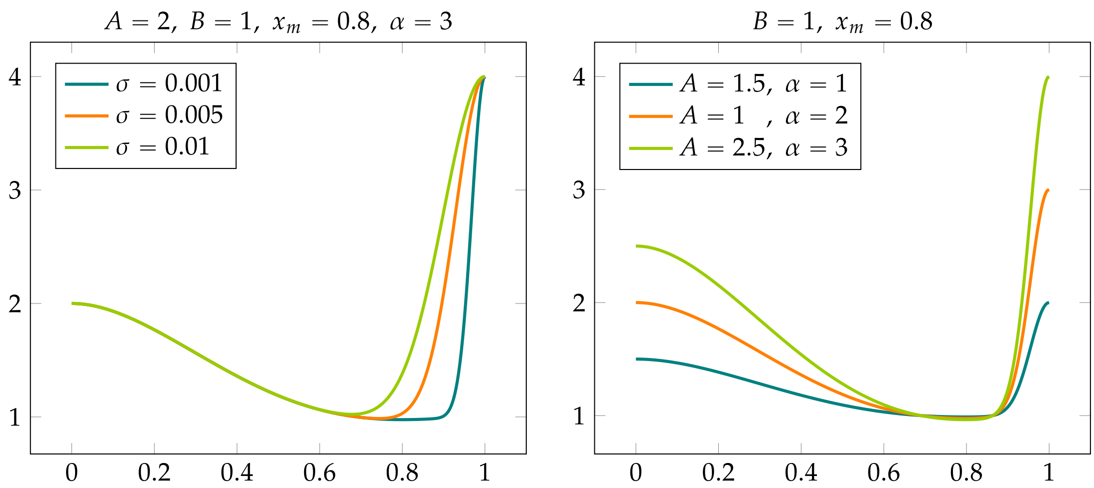

Figure 1: Overview of the polynomial approach with Gaussian correction. The top plot is meant to show the main features of the model, while the bottom two plots depict how the function $f(x)$ changes picking different values of the parameters.

$$\begin{cases} x_m < 0.5 \\ A < B \end{cases} \quad \text{or} \quad \begin{cases} x_m > 0.5 \\ A > B \end{cases} .$$

The two conditions are either contradicting property *(vi)* or property *(vii)* and this means that all requirements cannot be fulfilled with a simple polynomial of degree 4. At the same time, supposing to fulfill property *(vi)* and requiring a minimum at $x = x_m$, only property *(vii)* is violated, since $A > B$. This violation can be fixed adding a term that is almost 0 in the full domain $\mathcal{D}$ and which is increasing as fast as needed towards $x = 1$, without violating the requirement of arriving at the boundary with zero derivative. One possibility is to add a Gaussian correction,

$$g(x) = \alpha \exp\left[-\frac{(x-1)^2}{2\,\sigma}\right] \tag{3}$$

where α *corrects* the value at the boundary and the smallest σ the less will this Gaussian correction affect the behavior of $p(x)$.

Correcting eq. (1) using eq. (3), a function

$$f(x) \equiv p(x) + g(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \alpha \exp\left[-\frac{(x-1)^2}{2\,\sigma}\right] , \tag{4}$$

can be defined, where a_i are taken from eq. (2). Equation (4) leads to

$$f(1) = B + \alpha ,$$

which in turn accommodates property *(vii)*, if α is chosen sufficiently large, namely $\alpha > A - B$. It is worth remarking that the Gaussian correction in eq. (4) does not affect properties *(i)* and *(ii)*, while property *(iii)* has to be slightly adjusted. x_m has been imposed to be the minimum of $p(x)$, whereas $f(x)$ has a minimum for $x = \bar{x}_m$ and in general it is $\bar{x}_m < x_m$, with $x = x_m$ only in the limit $\sigma \to 0$, i.e. when the correction vanishes. The smaller is sigma the narrower will be the correction and the closer will be $\bar{x}_m$ to x_m. In fig. 1 a graphical overview of the model is given.

2 Combining arc-tangent functions

A slightly less systematic approach, which might be advantageous when it comes to choose the final values of parameters, is to build the model shaping known elementary functions, taking advantage of their features. Property *(ii)* and in general a horizontal plateau reminds immediately of the arc-tangent asymptotes. It should then not be hard to combine a couple of arc-tangent functions in a way such that the decrease between $x = 0$ and the minimum x_m is "dominated" by a $\arctan(-x)$-like part and the raise afterwords is imposed by a $\arctan(x)$-like one.

In order to adjust the shape of the left and right part, few shift and dilatation parameters are needed. Consider

$$g(x) \equiv A_L \arctan\big[S_L\,(F_L - x)\big] + A_R \arctan\big[S_R\,(x - F_R)\big] + \kappa , \tag{5}$$

where $A_{\{L,R\}}$, $S_{\{L,R\}}$ and $F_{\{L,R\}}$ represent the amplitude, slope and flex position of the left and right parts, respectively, while κ is just an overall vertical shift to let codomain of the function be in $\mathbb{R}^+$. Note that the flex points need to be within $\mathcal{D}$ and in particular $0 < F_L < F_R < 1$, while amplitudes and slopes are positive parameters.

Plotting eq. (5) for some values of the parameters can give an idea about how such a function behaves, but to choose the parameters appropriately it is worth studying few aspects of $g(x)$. First

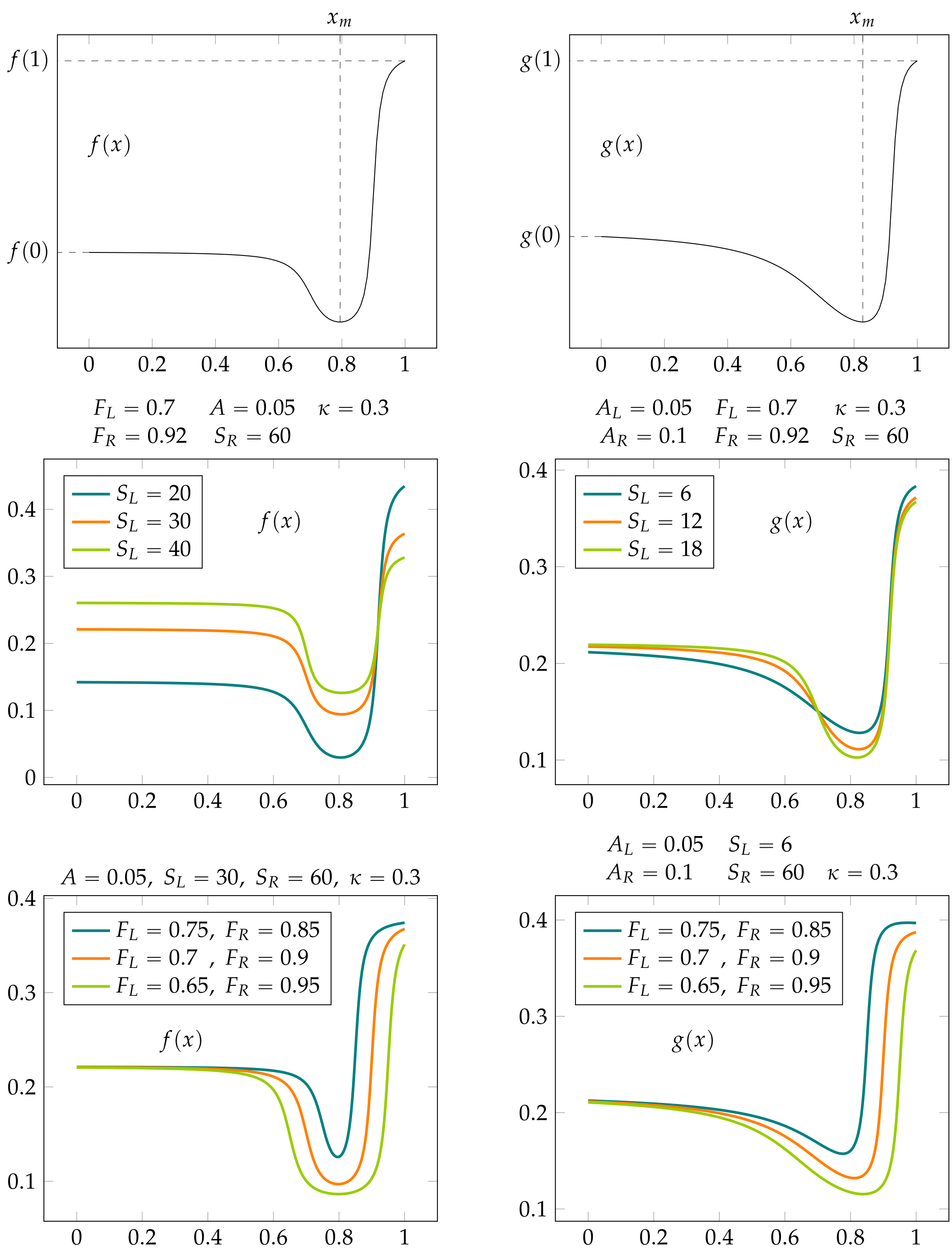

Figure 2: Overview of the arc-tangent approach. In the plots in the left column refer to the simplified model in which the function defined in eq. (7) has exactly one minimum. In the right column, instead, the initial model function defined in eq. (5) is plotted, choosing the parameters in a way such that only one extremum exist in $\mathcal{D}$. The effect of varying the left slope or the position of the flex points is shown.

of all, values at the boundaries can be evaluated,

$$g(0) = \kappa + A_L \arctan(F_L \cdot S_L) - A_R \arctan(F_R \cdot S_R)$$
$$g(1) = \kappa - A_L \arctan\big[(1 - F_L) \cdot S_L\big] + A_R \arctan\big[(1 - F_R) \cdot S_R\big] \ .$$

Comparing $g(0)$ and $g(1)$ and considering that $0 < F_L < F_R < 1$ implies $0 < (1-F_R) < (1-F_L) < 1$, it is easy to argue that for A_R sufficiently larger than A_L, it is $g(0) < g(1)$, i.e. property *(vii)* is fulfilled. The derivative of f reads

$$f'(x) = -\frac{A_L\, S_L}{1 + S_L^2\,(F_L - x)^2} + \frac{A_R\, S_R}{1 + S_R^2\,(F_R - x)^2} \tag{6}$$

and signals that the larger the slopes parameters are the better property *(ii)* is fulfilled. Furthermore, the closer the flex points are to 0 (or 1) the larger has to be the left (or right) slope parameter to make f flat at the boundaries of its domain. Equation (6) entails also the extrema of $g(x)$ which are in general two,

$$x_{1,2} = \frac{A_R\, F_L\, S_L - A_L\, F_R\, S_R}{A_R\, S_L - A_L\, S_R} - \sqrt{\frac{A_L\, A_R\, S_L^2 - (A_L^2 + A_R^2)\, S_L\, S_R + A_L\, A_R\,(1 + (F_L - F_R)^2\, S_L^2)\, S_R^2}{S_L\, S_R\,(A_R\, S_L - A_L\, S_R)^2)}}$$

and depending on the values of the parameters might both lie in $\mathcal{D}$. Indeed, we are adding a monotonically decreasing function and a monotonically increasing one and the interplay between their steepness will give raise to minima and/or maxima. It is interesting to note that, solving $f'(x) = 0$, the coefficient of x^2 is

$$A_R\, S_R\, S_L^2 - A_L\, S_L\, S_R^2 = S_R\, S_L\,(A_R\, S_L - A_L\, S_R)$$

and this means that, if

$$A_R\, S_L = A_L\, S_R \ , \tag{7}$$

the derivative has one root only,

$$x_m = \frac{S_L^2\,\big[1 + (F_R^2 - F_L^2)\, S_R^2\big] - S_R^2}{2\,(F_R - F_L) S_L^2 S_R^2} \tag{8}$$

which can be shown to be a minimum in our case[2]. Plugging in eq. (7) into eq. (5), we obtain our refined, final model,

$$f(x) \equiv A \left\{ \arctan\big[S_L\ (F_L - x)\big] + \frac{S_R}{S_L} \arctan\big[S_R\ (x - F_R)\big] \right\} + \kappa \ , \tag{9}$$

where we renamed the overall single amplitude factor simply into A. All remaining properties are satisfied and, of course, property *(vi)* requires an appropriate choice of the parameters. It is worth remarking that the shape of the function around the minimum depends on the slopes and the distance between the two flexes. The larger these quantities are, the broader will be the plateau around the minimum. This property is shown graphically in fig. 2 together with other aspects.

It is important to remark that $g(x)$ can, of course, be used instead of $f(x)$, provided that the parameters are carefully chosen, so that the second extremum falls outside the domain. A more difficult choice of the values leads to a finer shaping of the curve, which might be advantageous or even needed in some contexts.

[2]The sign of $f''(x_m)$ is that of $F_R - F_L$, which is positive by construction.

3 Modulating the cosine function

Another shaping, similar to that done in section 2, can be done on any function that has a maximum for $x = 0$ and $x = 1$, i.e. the boundary of our domain. Among the elementary functions, $\cos(2\,\pi\,x)$ displays exactly this behaviour and, on top, it already fulfils properties *(i)* to *(v)*. However, properties *(vi)* and *(vii)* are not satisfied, since $\cos(0) = \cos(2\,\pi)$ and its minimum is at $x = 1/2$.

The main idea to shape the cosine function according to our needs is to adjust it by a function that "stretches it more at $x = 1$ than at $x = 0$". In principle, a monotonically increasing function would act as desired, but property *(ii)* must not be destroyed by the modulation and this implies that the modulating function as well should have zero derivative approaching $x = 0$ and $x = 1$, as well as not invert the cosine slope close to these two boundaries. For example, choosing a linear modulation would make the function loose properties *(ii)* and *(iii)* and a cleverer choice like multiplying the cosine by $\alpha\,\sqrt{(a\,x)^2 + \varepsilon^2}$ would imply to have to accept to approach $x = 1$ with small but finite derivative. Furthermore, the further needed adjustment to impose property *(vi)* would make the choice of the parameters very though if not impossible.

A different approach that gives more flexibility is to "stretch" only one part of the cosine function, namely between $x = x_m$ and $x = 1$. If this is done with a monotonically increasing term that approaches $x = x_m$ as well as $x = 1$ with zero derivative, property *(vii)* stays fulfilled without violating any other. Since the similar the new term is to the initial function in $[x_m, 1]$ the better, it is ideal to use the same initial function. Before discussing the model quantitatively, it is important to notice that the position of x_m is not affected by the stretching and another adjustment is needed to fulfil property *(vi)*, too. To shift the minimum without invalidating the previous efforts, it is possible to simply consider the cosine function of a power of x, which shifts the (first) minimum to the right.

Consider then

$$f(x) \equiv \underbrace{\alpha\cos(2\,\pi\,x^n)}_{g(x)} + \beta\left[\cos(2\,\pi\,x^n) + 1\right]\cdot\Theta(x - x_m) + \kappa\,, \tag{10}$$

where α and β are two positive amplitude parameters, $n \geqslant 1$ is another parameter to adjust the minimum position, x_m is the minimum of $g(x)$, κ an overall constant to shift vertically the function and

$$\Theta(x - x_m) \equiv \begin{cases} 0 & \text{if} \quad x \leqslant x_m \\ 1 & \text{if} \quad x > x_m \end{cases}$$

is the Heaviside function **defined to be zero at** $x = x_m$. It is easy to show that

$$g'(x) = -2\,\pi\,\alpha\,n\,x^{n-1}\,\sin(2\,\pi\,x^n) \quad \Rightarrow \quad x_m = \left(\frac{1}{2}\right)^{\frac{1}{n}}\,,$$

which make eq. (10) explicit as

$$f(x) \equiv \alpha\cos(2\,\pi\,x^n) + \beta\left[\cos(2\,\pi\,x^n) + 1\right]\cdot\Theta\Big(x - 2^{-\frac{1}{n}}\Big) + \kappa\,. \tag{11}$$

The values at the boundaries are $f(0) = \alpha + \kappa$ and $f(1) = \alpha + 2\,\beta + \kappa$, which implies $f(0) < f(1)$ since $\beta > 0$. Property *(vi)* is fulfilled if and only if

$$n \approx -\frac{1}{\log_2 0.8} \simeq 3.106$$

and in fig. 3 it can be seen how the shape of $f(x)$ varying n together with other properties.

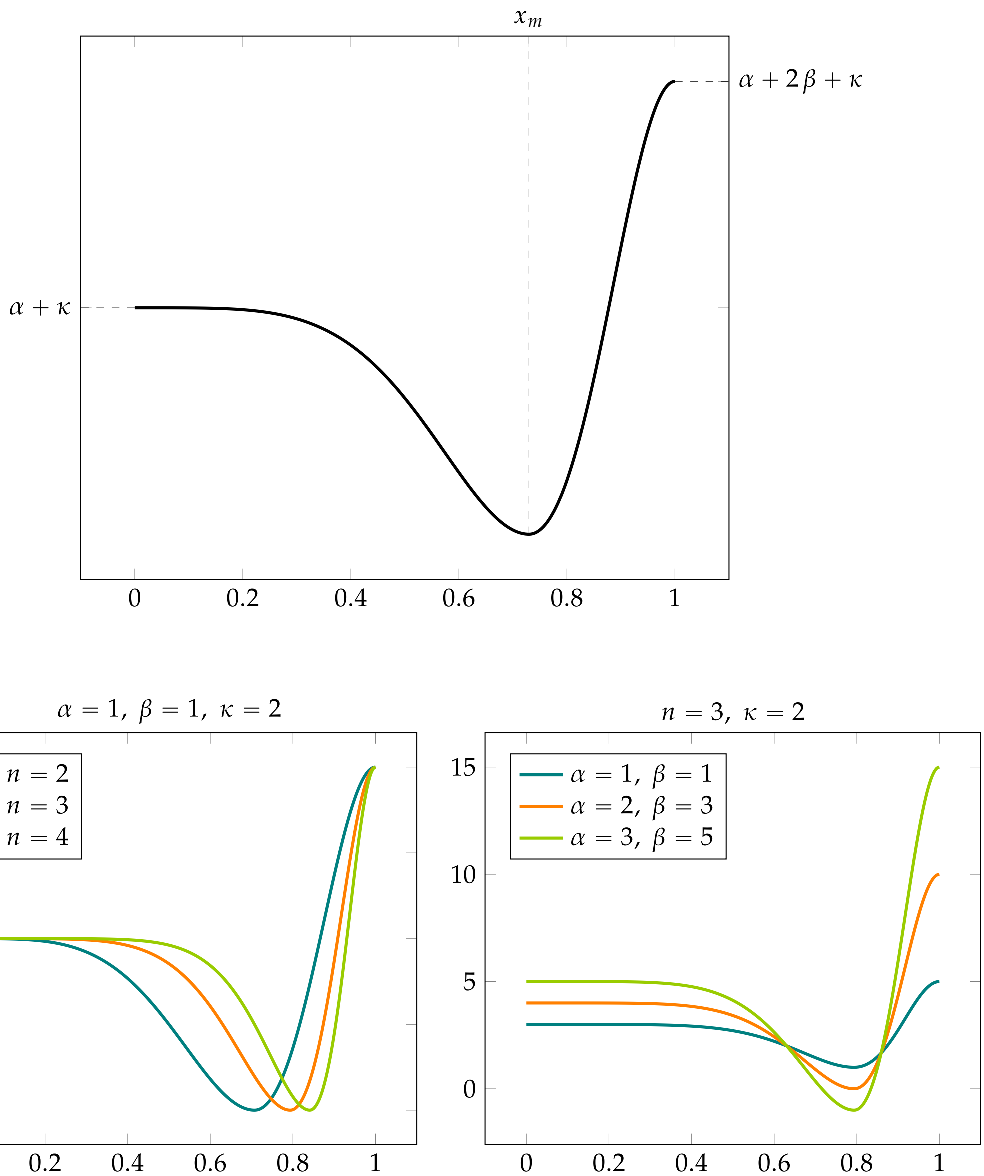

Figure 3: Overview of the cosine-based model. The top plot is meant to show the main features of the model, while the bottom two plots depict how the function $f(x)$ changes picking different values of the parameters.

Changing parameters in the models

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In each model few parameters with different meaning have been introduced, always describing a given feature of the model. However, for later tuning of the curve, the most convenient choice is to use the following quantities:

- the value y_0 of the function at $x = 0$;
- the value y_1 of the function at $x = 1$;
- the position x_m of the minimum of the function[3] and
- the value y_m of the function at $x = x_m$.

4 Rewriting the cosine model

The cosine model described in section 3 is the easiest to be rewritten in terms of y_0, y_1, x_m and y_m. The parameters of the model are α, β, κ and n, i.e. as many as the new ones. From eq. (11) and its derivative it is immediate to write down the following system of equations,

$$\begin{cases} y_0 = \alpha + \kappa \\ y_1 = \alpha + 2\beta + k \\ x_m = 2^{-\frac{1}{n}} \\ y_m = -\alpha + \kappa \end{cases} \tag{12}$$

which, solved for the model parameters, leads to

$$\begin{cases} \alpha = \dfrac{1}{2}\,(y_0 - y_m) \\ \beta = \dfrac{1}{2}\,(y_1 - y_0) \\ \kappa = \dfrac{1}{2}\,(y_0 + y_m) \\ n = -\dfrac{1}{\log_2 x_m} \end{cases} . \tag{13}$$

Inserting eq. (13) into eq. (11) and defining $X \equiv 2\,\pi\,x^{-\frac{1}{\log_2 x_m}}$ to shorten notation leads to

$$f(x) = \frac{1}{2}\,(y_0 - y_m)\,\cos(X) + \frac{1}{2}\,(y_1 - y_0)\,\big[\cos(X) + 1\big]\cdot\Theta(x - x_m) + \frac{1}{2}\,(y_0 + y_m)\,, \tag{14}$$

which can be rewritten as

$$\begin{aligned} f(x) = {} & \frac{1}{2}\,y_0\,\big[1 + \cos(X)\big]\cdot\big[1 - \Theta\big(x - x_m\big)\big] \\ & + \frac{1}{2}\,y_1\,\big[1 + \cos(X)\big]\cdot\Theta\big(x - x_m\big) + \frac{1}{2}\,y_m\,\big[1 - \cos(X)\big]\,, \end{aligned} \tag{15}$$

namely making more explicit the meaning of the new parameters y_0, y_1 and y_m.

[3] The symbol x_m has been previously used, but here it is to be regarded as a completely new parameter.

5 Remark about the other models

The polynomial model introduced in section 1 should in principle allow for a similar rewriting, since also there exactly *four* parameters were introduced: A, B, α and σ. However, because of the Gaussian correction in eq. (4), it is not possible to analytically locate the position of the minimum of $f(x)$ and, hence, a system of equations cannot be explicitly written. Given the values of the four new parameters y_0, y_1, x_m and y_m, it would still be possible to set up a system of equations, which however is not linear and most likely to be solved numerically. All in all, for the polynomial case, it is not possible to write down an expression analogous to eq. (14).

Lastly, in the arctangent model described in section 2, at least six parameters were introduced and only a partial rewriting in terms of the new one could be attempted.